Frequency Domain Model Validation in Wasserstein Metric

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Motivation: Density based Model Validation

- True (but unknown) dynamics: Lorenz system with parameters
  \( \sigma = 10, \beta = \frac{8}{3}, \rho = 2 \)

  \[
  \begin{align*}
  \dot{x}_1 &= \sigma (x_2 - x_1) \\
  \dot{x}_2 &= x_1 (\rho - x_3) - x_2 \\
  \dot{x}_3 &= x_1 x_2 - \beta x_3
  \end{align*}
  \]

- Model: linearization about the fixed point
  \( (x^*, y^*, z^*) = \left( \sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1 \right) \)
Nonlinear model validation depends on the choice of $x_0$

- True with $x_0 = (0, 1, 1.05)$, True with $x_0 = (0, 1, 1.050001)$
Nonlinear model validation depends on the choice of $x_0$

- True with $x_0 = (0, 1, 1.05)$, True with $x_0 = (0, 1, 1.050001)$
- Model with $x_0 = (0, 1, 1.05)$, Model with $x_0 = (0, 1, 1.050001)$
Nonlinear model validation depends on the choice of $\rho$

- True with $\rho = 14$, True with $\rho = 28$, fixed $x_0 = (0, 1, 1.05)$
Motivation: Density based Model Validation

\[ \dot{x}_1 = -x_2, \quad \dot{x}_2 = \sin x_1, \]

\[ y = x = (x_1, x_2)^\top. \]

Compare trajectories densities
Summary of the proposed Model Validation formulation


Halder, and Bhattacharya, “Further Results on Probabilistic Model Validation in Wasserstein Metric”, CDC 2012.
Summary of the proposed Model Validation formulation

Question: does this have a systems-theoretic interpretation?

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Consider two stable LTI systems with transfer functions (matrices) $G$ and $\hat{G}$, excited by Gaussian white noise $u(t) \sim \mathcal{N}(0, \text{diag}(\sigma_u^2))$, then

1. **SISO and MISO:**
   \[
   W_\infty (G, \hat{G}) = \sqrt{2\pi \sigma_u} \left| \|G(j\omega)\|_2 - \|\hat{G}(j\omega)\|_2 \right|
   \]

2. **MIMO:**
   \[
   W_\infty (G, \hat{G}) = \sqrt{2\pi \sigma_u} \left( \|G(j\omega)\|_2^2 + \|\hat{G}(j\omega)\|_2^2 \right) - 2 \text{tr} \left[ \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} G^H(j\omega) G(j\omega) d\omega \right)^{1/2} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{G}^H(j\omega) \hat{G}(j\omega) d\omega \right)^{1/2} \right]^{1/2}
   \]
Observation: the "green gap" $\to 0$, if $[\Sigma_\infty, \hat{\Sigma}_\infty] \to 0$.

(Open problem: when does an algebraic Riccati pair commute?)

Question: how to normalize the Wasserstein metric?

Question: can we compare it with existing metrics?

Bounds for MIMO $W_\infty$
Observation: the “green gap” → 0, if $\left[ \Sigma_\infty, \hat{\Sigma}_\infty \right] \to 0$.

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**Question:** how to normalize the Wasserstein metric?

**Question:** can we compare it with existing metrics?
Geometric Meaning & Intrinsic Normalization of SISO $\mathcal{W}_\infty$

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Comparing $W_\infty$ and $\delta_\nu := \sup_\omega \kappa(\omega)$

- **Un-normalized comparison on Complex plane:**
  \[
  \sup_\omega \kappa^{\text{proj}}(\omega) \geq W_\infty
  \]

- **Normalized comparison on Riemann sphere:**
  \[
  \overline{W}_S \left( G, \hat{G} \right) = \frac{2}{\pi} \left| \arctan \|G\|_2 - \arctan \|\hat{G}\|_2 \right|,
  \text{compare } \overline{W}_S \text{ with } \delta_\nu
  \]
Summary

- Frequency-domain formula for LTI model validation with stochastic forcing
- Intrinsic normalization
- Comparison with $\nu$-gap metric for single output case

Future work

- LTI validation between minimum and non-minimum phase systems
- Systems-theoretic interpretation for nonlinear systems
- Closed-loop model validation
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Thank you.