Frobenius-Perron Operator
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Frobenius-Perron Operator

Linear Operator $\mathcal{P}_t$

**Given dynamics**

$$\dot{x} = F(t,x),$$

with $p(t_0, x)$ as the initial state density function.

- Evolution of density is given by

$$p(t, x) := \mathcal{P}_t p(t_0, x).$$

- $\mathcal{P}_t$ has following properties

  $$\mathcal{P}_t (\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 \mathcal{P}_t p_1 + \lambda_2 \mathcal{P}_t p_2$$  linearity

  $$\mathcal{P}_t p \geq 0 \text{ if } p \geq 0,$$  positivity

  $$\int_{\mathcal{X}} \mathcal{P}_t p(t_0, x) \mu(dx) = \int_{\mathcal{X}} p(t_0, x) \mu(dx)$$  measure preserving

$\mathcal{P}_t$ is defined by

$$\frac{\partial p}{\partial t} + \nabla \cdot (p F) = 0$$

- Continuity equation
- FPK without diffusion term
- First order linear PDE
First Order PDEs

Method of Characteristics

\[
\frac{\partial p}{\partial t} + \nabla \cdot (pF) = \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial pF_i(t, \mathbf{x})}{\partial x_i} = 0 \]

This is of the form

\[ a(t, \mathbf{x}, p)p_t + \sum_i b_i(t, \mathbf{x}, p)p_{x_i} = c(t, \mathbf{x}, p). \]

Lagrange-Charpit equations

\[
\frac{dt}{a(t, \mathbf{x}, p)} = \frac{dx_i}{b_i(t, \mathbf{x}, p)} = \frac{dp}{c(t, \mathbf{x}, p)}
\]
Uncertainty Propagation with Frobenius-Perron Operator

Characteristic Equations

Lagrange-Charpit equations

\[
\frac{dt}{a(t, \mathbf{x}, p)} = \frac{dx_i}{b_i(t, \mathbf{x}, p)} = \frac{dp}{c(t, \mathbf{x}, p)}
\]

- Let \( s \) be parameterization of characteristic curves
- Characteristic curves are given by the ODEs

\[
\frac{dt}{ds} = a(t, \mathbf{x}, p)
\]

\[
\frac{dx_i}{ds} = b_i(t, \mathbf{x}, p)
\]

\[
\frac{dp}{ds} = c(t, \mathbf{x}, p)
\]
Solution of Continuity Equation

For continuity equation

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(t, \mathbf{x}) + p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i} = 0$$

$$a(t, \mathbf{x}, p) = 1, \quad b_i(t, \mathbf{x}, p) = F_i(t, \mathbf{x}), \quad c(t, \mathbf{x}, p) = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}.$$

Characteristic equations

$$\frac{dt}{ds} = 1$$

$$\frac{dx_i}{ds} = F_i(t, \mathbf{x})$$

$$\frac{dp}{ds} = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}$$

$$\dot{x} = F(t, \mathbf{x}) \quad \text{evolution of } x(t)$$

$$\dot{p} = -p(\nabla \cdot F) \quad \text{evolution of } p \text{ along } x(t)$$

Initial Conditions

- $\mathbf{x}_0$ Samples from $p(t_0, \mathbf{x})$
- $p_0 = p(t_0, \mathbf{x}_0)$ Values of $p(t_0, \mathbf{x})$ at $\mathbf{x}_0$
Parametric Uncertainty & Process Noise

Given system dynamics

\[ \dot{x} = F(t, x, \Delta) + n(t, \omega) \]

- Expand \( n(t, \omega) \) using KL expansion.
- New parameters: \( \xi := (\xi_0, \xi_0^*, \cdots, \xi_N, \xi_N^*)^T \)
- PDF: \( p_\xi(\xi) \)
- Parameter PDF: \( p_\Delta(\Delta) \)
- State IC PDF: \( p_x(t_0, x) \)

Augment state space

\[
X := \begin{pmatrix} x \\ \Delta \\ \xi \end{pmatrix}, \quad \text{with} \quad \dot{X} := \begin{pmatrix} G(t, x, \Delta, \xi) \\ 0 \\ 0 \end{pmatrix} = H(t, X)
\]

with \( p_X(t_0, X) := p_x(t_0, x)p_\Delta(\Delta)p_\xi(\xi) \) and \( p_X(t, X) := \mathcal{P}_t p_X(t_0, X) \).
Better Accuracy & Faster Convergence than MC

(a) First moment
- Data generated from univariate normal distribution
- **MC**: PDF from kernel density estimation
- **FP**: PDF from spline interpolation
- Samples generated 1000 times for a given size. Plots show average error vs sample size

(b) Second moment

Requires \( \frac{\partial F_i(x)}{\partial x_i} \).
Nonlinear Example

3 DOF Vinh’s Equation Models motion of spacecraft during planetary entry

\[ \begin{align*}
\dot{h} &= V \sin(\gamma) \\
\dot{V} &= -\frac{\rho R_0}{2 B_c} V^2 - \frac{g R_0}{v_c^2} \sin(\gamma) \\
\dot{\gamma} &= \frac{\rho R_0}{2 B_c} \frac{C_L}{C_D} V + \frac{g R_0}{v_c^2} \cos(\gamma) \left( \frac{V}{R_0 + h} - \frac{1}{V} \right).
\end{align*} \]

- \( R_0 \) – radius of Mars
- \( \rho \) – atmospheric density
- \( v_c \) – escape velocity
- \( \frac{C_L}{C_D} \) – lift over drag
- \( B_c \) – ballistic coefficient
- \( h \) – height
- \( V \) – velocity
- \( \gamma \) – flight path angle
3DOF Vinh’s Equation

- Gaussian initial condition uncertainty in $(h, V, \gamma)$
Frobenius-Perron Operator

Papers

